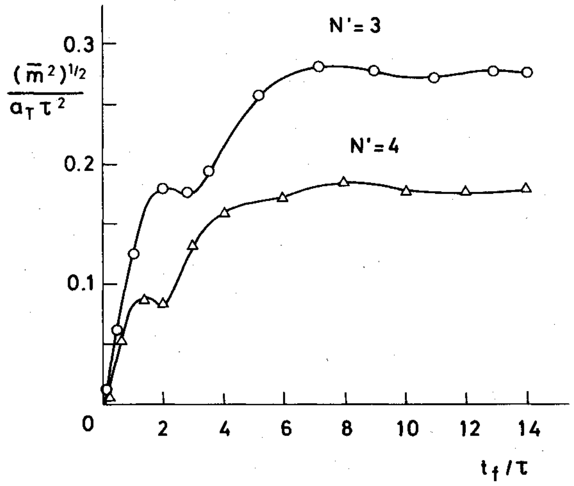

Fig. 2 Normalized average (rms) miss distance.

for a first-order guidance transfer function

$$F(s)=1/(1+\tau s) \tag{22}$$

is shown as a function of normalized time of flight in Fig. 2. The results were compared to the average of 50 Monte Carlo runs using the actual maneuver process, showing less than 1% of difference for short time of flight. Results for large values of t_f were identical to previously published data.

Conclusions

The direct method presented in this paper for the calculation of the mean square ensemble average of nonstationary functions can be of great use in the analysis of linear time-varying systems with periodical random phase inputs. The results can be obtained either by a closed-form solution or by a single computer run instead of numerous Monte Carlo simulations. The method can be equally applied for adjoint analysis.

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Recursive Parameter Identification for Nonlinear Stochastic Processes

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Introduction

SEVERAL methods[1-7] have been presented for obtaining least-squares estimates of unknown parameters of systems modeled by nonlinear differential equations with discrete measurements made on their response. The unidentified parameters are estimated by minimizing a mean square performance index. Convergence of the iterative algorithms of quasilinearization of Bellman,[2] the Newtonian iteration procedure of Goodman,[1] the parametric differentiation of Chapman and Kirk,[3] and the continuation method of Wasserstrom[4] are all dependent on a good initial guess of the parametric vector. Although convergence and the initial estimate of the parametric vector are closely related for any iterative procedure, choice of the computational algorithm is of paramount importance in affecting the rate of convergence. In the present analysis the nonlinear process and observation equations are linearized and cast into standard linear forms in terms of state, parametric, and observation difference vectors. The Kalman filter algorithm is then employed to obtain recursive estimates of the state difference and parametric difference vectors. The recursive estimate algorithm of the state and parametric vectors is then derived.

Analysis

Consider a nonlinear discrete-time stochastic process represented by

$$x_k=\phi(x_{k-1},\xi,u_{k-1})+w_k \qquad w_k\sim N(0,Q_k) \tag{1a}$$

$$z_k=h(x_k)+v_k \qquad v_k\sim N(0,R_k) \tag{1b}$$

where x is an n-dimensional state vector, u is an m-dimensional control vector, z is a p-dimensional observation vector, and ξ is the true r-dimensional parametric vector which is not known a priori. w and v are the process and measurement noise vectors, respectively.

Denoting the estimate of the parametric vector at step $(k-1)$ by ξ_{k-1}, and its estimate error by γ_{k-1}, Eq. (1a) may be written in the form

$$x_k=\phi(x_{k-1},\xi_{k-1},u_{k-1})+\Pi_{k-1}\gamma_{k-1}+w_k \tag{2}$$

where

$$\Pi_{k-1}=(\partial\phi/\partial\xi)_{k-1} \qquad \gamma_{k-1}=\xi-\xi_{k-1} \tag{3}$$

Using the differencing approach,[8] Eq. (2) may be cast into the convenient linear form

$$s_k=\Phi_{k-2}s_{k-1}+(\Pi_{k-1}-\Pi_{k-2})\gamma_{k-1}+U_{k-2}c_{k-1}+\nu_k \tag{4}$$

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where

$$\Phi_{k-2}=\left(\frac{\partial\phi}{\partial x}\right)_{k-2}\quad \Pi_{k-2}=\left(\frac{\partial\phi}{\partial\xi}\right)_{k-2}\quad U_{k-2}=\left(\frac{\partial\phi}{\partial u}\right)_{k-2} \tag{5a,b,c}$$

$$s_{k-1}=x_{k-1}-x_{k-2}\qquad c_{k-1}=u_{k-1}-u_{k-2} \tag{5d,e}$$

$$\nu_k=w_k-w_{k-1}\sim N[0,(Q_k+Q_{k-1})] \tag{5f}$$

Introducing the measurements difference vector

$$y_k=z_k-z_{k-1} \tag{6}$$

the measurement difference equation takes the form[6]

$$y_k=H_{k-1}s_k+\mu_k \tag{7}$$

where

$$\mu_k=v_k-v_{k-1}\sim N[0,(R_k+R_{k-1})] \tag{8}$$

In the present work, our main interest is to estimate in a recursive manner the undefined parametric vector together with the system state. The estimator algorithm will be convergent if the estimation error decreases monotonically. Such behavior suggests the following relation

$$\gamma_k=\Psi\gamma_{k-1}\qquad \Psi=\mathrm{diag}\,\{e^{\lambda_1\tau}e^{\lambda_2\tau}\ldots e^{\lambda_r\tau}\} \tag{9}$$

where λ_i; $i=1,\ldots,r$ are eigenvalues to yield the desired convergence rate and τ is the sample time. Introducing the augmented $(n+r)$ dimensional state vector

$$\eta_k^T=(s_k^T\ \vdots\ \gamma_k^T) \tag{10}$$

From Eqs. (4,7,9, and 10) the augmented system may be written as

$$\eta_k=A_{k-2}\eta_{k-1}+B_{k-2}c_{k-1}+L\nu_k\qquad y_k=D_{k-1}\eta_k+\mu_k \tag{11}$$

where

$$A_{k-2}=\left[\begin{array}{c|c}\Phi_{k-2} & (\Pi_{k-1}-\Pi_{k-2})\\ \hline 0 & \Psi\end{array}\right]\qquad B_{k-2}=\left[\begin{array}{c}U_{k-2}\\ \hline 0\end{array}\right] \tag{12a,b}$$

$$L=\left[\begin{array}{c}I_n\\ \hline 0\end{array}\right]\qquad D_{k-1}=[H_{k-1}\ \vdots\ 0] \tag{12c,d}$$

I_n is an n-dimensional identity matrix. Equations (11) have the standard form of a linear discrete-time process with noise-corrupted observations. Employment of the Kalman filter algorithm gives the optimal estimator equations

$$P_k^*(-)=A_{k-2}P_{k-1}^*(+)A_{k-2}^T+L(Q_{k-1}+Q_k)L^T \tag{13a}$$

$$\hat\eta_k(+)=A_{k-2}\hat\eta_{k-1}(+)+B_{k-2}c_{k-1}+K_k^*[y_k-D_{k-1}\hat\eta_k(-)] \tag{13b}$$

$$P_k^*(+)=[I-K_k^*D_{k-1}]P_k^*(-) \tag{13c}$$

$$K_k^*=P_k^*(-)D_{k-1}^T[D_{k-1}P_k^*(-)D_{k-1}^T+(R_k+R_{k-1})]^{-1} \tag{13d}$$

In the above equations, P^* is the augmented state difference error covariance matrix which should be initialized in order to be able to manipulate Eqs. (13). It is defined by

$$P_k^*=\mathrm{E}[\tilde\eta_k\tilde\eta_k^T] \tag{14}$$

where E $(\cdot)$ is the expectation and

$$\tilde\eta_k=\hat\eta_k-\eta_k \tag{15}$$

Based upon Eq. (10), P_k^* takes the form

$$P_k^*=\left[\begin{array}{c|c}\mathrm{E}(\tilde s_k\tilde s_k^T) & \mathrm{E}(\tilde s_k\gamma_k^T)\\ \hline \mathrm{E}(\gamma_k\tilde s_k^T) & \mathrm{E}(\gamma_k\gamma_k^T)\end{array}\right] \tag{16}$$

Introducing the covariance matrices

$$\Sigma_k^1=\mathrm{E}(\tilde s_k\gamma_k^T)\qquad \Sigma_k^2=\mathrm{E}(\gamma_k\tilde s_k^T)$$
$$\Gamma_k=\mathrm{E}(\gamma_k\gamma_k^T)\qquad S_k=\mathrm{E}(\tilde s_k\tilde s_k^T) \tag{17}$$

the augmented state difference error covariance matrix P_k^* may be written as

$$P_k^*=\left[\begin{array}{c|c}S_k & \Sigma_k^1\\ \hline \Sigma_k^2 & \Gamma_k\end{array}\right] \tag{18}$$

The system state estimate propagation during the sample time interval is obtained from Eq. (2) as

$$\hat x_k(-)=\phi[\hat x_{k-1}(+),\xi_{k-1},u_{k-1}]+\Pi_{k-1}\hat\gamma_{k-1}(+) \tag{19}$$

On the other hand Eq. (2) may be written as

$$x_k=\phi[\hat x_{k-1}(+),\xi_{k-1},u_{k-1}]$$
$$-\Phi_{k-1}\tilde x_{k-1}(+)+\Pi_{k-1}\gamma_{k-1}+w_k \tag{20}$$

where

$$\tilde x_{k-1}(+)=\hat x_{k-1}(+)-x_{k-1}\qquad \Phi_{k-1}=(\partial\phi/\partial x)\hat x_{k-1}(+) \tag{21a,b}$$

Subtracting Eq. (20) from Eq. (19) and making use of Eq. (21a) yields

$$\tilde x_k(-)=\Phi_{k-1}\tilde x_{k-1}(+)-w_k \tag{22}$$

In deriving the above equation, it was assumed that $\hat\gamma(+)\to\hat\gamma$.

Noting that a relation similar to that given by Eq. (21a) holds for the vector s, Eqs. (5d) and (21a) give

$$\tilde s_k=\tilde x_k-\tilde x_{k-1} \tag{23}$$

After some lengthy algebraic manipulation, the initial augmented state difference error covariance matrix may be obtained from Eqs. (17, and 21-23) as

$$P_2^*=\left[\begin{array}{c|c}S_2 & \Sigma_2^1\\ \hline \Sigma_2^2 & \Gamma_2\end{array}\right] \tag{24}$$

where

$$S_2=(\Phi_1-I)P_1(\Phi_1^T-I)+Q_2$$
$$\Sigma_2^1=[\phi_0\Sigma_1^1-(\Pi_1-\Pi_0)\Gamma_1]\Psi^T$$
$$\Sigma_2^2=\Psi[\Sigma_1^2\Phi_0^T-\Gamma_1(\Pi_1-\Pi_0)^T] \tag{25}$$

Initial correlation matrices and initial Γ are to be assumed. Simple sensitivity analysis may serve as a guideline for assuming Σ_0^1, Σ_0^2, and Γ_0.

It is seen from the above equations that the updating process starts at $k=3$. Further manipulations are needed in order to write Eqs. (13) in forms suitable for computation. Based upon Eq. (18), Eq. (13) may be decomposed into four matrix equations as

$$S_k(-)=\Phi_{k-2}S_{k-1}(+)\Phi_{k-2}^T+(\Pi_{k-1}-\Pi_{k-2})$$
$$\cdot\Sigma_{k-1}^2(+)\Phi_{k-2}^T+\Phi_{k-2}\Sigma_{k-1}^1(+)(\Pi_{k-1}-\Pi_{k-2})^T$$
$$+(\Pi_{k-1}-\Pi_{k-2})\Gamma_{k-1}(+)(\Pi_{k-1}-\Pi_{k-2})^T$$
$$+Q_{k-1}+Q_k$$
$$\Sigma_k^1(-)=\Phi_{k-2}\Sigma_{k-1}^1(+)\Psi^T+(\Pi_{k-1}-\Pi_{k-2})\Gamma_{k-1}(+)\Psi^T$$
$$\Sigma_k^2(-)=\Psi\Sigma_{k-1}^2(+)\Phi_{k-2}^T+\Psi\Gamma_{k-1}(+)(\Pi_{k-1}-\Pi_{k-2})^T$$
$$\Gamma_k(-)=\Psi\Gamma_{k-1}(+)\Psi^T \tag{26}$$

The Kalman gain matrix given by Eq. (13d) may be decomposed into the two matrix equations

$$K_k^{s}=S_k(-)H_{k-1}^T[H_{k-1}S_k(-)H_{k-1}^T+R_k+R_{k-1}]^{-1}$$
$$K_k^{\gamma}=\Sigma_k^2(-)H_{k-1}^T \tag{27}$$

where

$$K_k=\underset{p}{\begin{pmatrix}K_k^s\\ K_k^\gamma\end{pmatrix}}\begin{matrix}\}n\\ \}r\end{matrix} \tag{28}$$

Similarly, with the help of Eqs. (12d), (18), and (28), Eq. (13c) may be decomposed as

$$S_k(+)=(I-K_k^sH_{k-1})S_k(-)$$
$$\Sigma_k^1(+)=(I-K_k^sH_{k-1})\Sigma_k^1(-)$$
$$\Sigma_k^2(+)=-K_k^\gamma H_{k-1}S_k(-)+\Sigma_k^2(-)$$
$$\Gamma_k(+)=\Gamma_k(-)-K_k^\gamma H_{k-1}\Sigma_k^1(-) \tag{29}$$

The augmented state estimate updating given by Eq. (22) may also be written in the form

$$\begin{pmatrix}\hat{S}_k(+)\\ \hat{\gamma}_k(+)\end{pmatrix}$$
$$=\begin{pmatrix}\Phi_{k-1}\hat{S}_{k-1}(+)(\Pi_{k-1}-\Pi_{k-2})\hat{\gamma}_{k-1}(+)\\ \Psi\hat{\gamma}_{k-1}(+)\end{pmatrix}+\begin{pmatrix}U_{k-2}\\ 0\end{pmatrix}c_{k-1}$$
$$+\begin{pmatrix}K_k^s\\ K_k^\gamma\end{pmatrix}\left\{z_k-z_{k-1}-[H_{k-1}\ \vdots\ 0]\begin{pmatrix}\hat{s}_k(-)\\ \hat{\gamma}_k(-)\end{pmatrix}\right\} \tag{30}$$

It is the state vector estimate and the parametric vector estimate that we are interested in rather than their differences. Making use of Eqs. (5d) and (5e), Eq. (30) may be written more conveniently as

$$\hat{x}_k(+)=(I+\Phi_{k-1}+K_k^sH_{k-1})\hat{x}_{k-1}(+)-K_k^sH_{k-1}\hat{x}_k(-)$$
$$+(\Pi_{k-1}-\Pi_{k-2})\hat{\gamma}_{k-1}(+)+U_{k-2}(u_{k-1}-u_{k-2})$$
$$+K_k^s(z_k-z_{k-1})$$

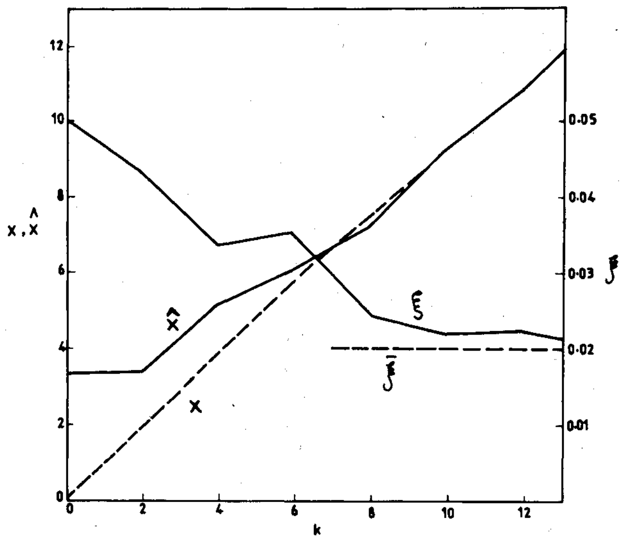

Fig. 1 State and parameter estimates of given example.

$$\hat{\gamma}_k(+)=\Psi\hat{\gamma}_{k-1}(+)$$
$$+K_k^\gamma[z_k-z_{k-1}-H_{k-1}\hat{x}_k(-)+H_{k-1}\hat{x}_{k-1}(-)]$$
$$\xi_k=\xi_{k-1}-[\hat{\gamma}_k(+)-\hat{\gamma}_{k-1}(+)] \tag{31}$$

Equations (26), (27), (29), and (31) provide the computational algorithm for simultaneous estimation of the state and parametric vectors. Since the Kalman filter algorithm (not the EKF) was employed, the stability of the estimation process is mainly that of the Kalman filter. For neutrally stable systems with nondisturbable or poorly disturbable states, the estimator diverges. On the other hand, the correlation matrices Σ^1 and Σ^2 that resulted due to the introduction of the parametric error convergence tuning matrix Ψ play an important role in the updating process of the parametric vector identification. Their existence provide the mechanism for feeding back the observation information to update the parametric vector estimate which in its turn enhances the parameter identification stability.

Application

The present approach is tested by considering the problem of identifying the constant parameter ξ and the state of the scalar system

$$x_k=x_{k-1}+1-0.1\xi x_{k-1}^2$$

with the observation given by

$$z_k=x_k+v_k \qquad v_k\sim N(0,r)$$

Observations were generated by digital simulation of the state equation for $x_0=0$ and $\xi=0.02$ with measurement noise variance $r=0.01$. The initial state error, correlation, and parameter error variances are $p_1=10$, $\sigma_1^1=\sigma_1^2=3$, and $\Gamma_1=5$. The state and parameter estimates are shown in Fig. 1.

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Orbital Decay Due to Drag in an Exponentially Varying Atmosphere

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Introduction

In order to illustrate the use of multiple variable expansions in satellite problems, Kevorkian[1] studied the idealized problem of the orbital decay of a satellite with constant aerodynamic coefficients in a thin constant-density atmosphere. The assumption of a constant density, which is admittedly physically unrealistic, leads to considerable simplification. It implies that the order of magnitude of the aerodynamic forces remains unchanged as the orbit evolves. Consequently, one can account for the small but cumulative aerodynamic perturbations for all times by means of one uniformly valid multivariable expansion. In the more realistic problem modeled by an exponentially varying density, this is no longer the case. A glide trajectory which originates at a sufficiently high altitude will remain only for a limited time in a regime (outer region) where the aerodynamic forces are small in comparison with the gravitational attraction. Eventually, as the orbit decays due to drag, the relative orders of magnitude of the aerodynamic vs gravitational terms will reverse (inner region). This type of problem requires the matching of an outer and inner expansion. Examples are studied in Refs. 2-4.

If the initial conditions are such that the glider completes a large number of orbits before entry into the inner region, one must still use a multiple variable expansion (or an equivalent formulation) to describe the outer solution uniformly, even though its duration is limited.

In this Note, we consider the simple problem of extending Kevorkian's results to an exponentially varying atmosphere in the outer region. We do not consider the solution in the inner region.

Mathematical Model

A planar motion is considered using the polar coordinates (R,θ) as follows[1]:

$$m\frac{\mathrm{d}^2R}{\mathrm{d}T^2}-mR\left(\frac{\mathrm{d}\theta}{\mathrm{d}T}\right)^2=-\frac{GmM}{R^2}-D\sin\gamma \tag{1}$$

$$mR\frac{\mathrm{d}^2\theta}{\mathrm{d}T^2}+2m\frac{\mathrm{d}R}{\mathrm{d}T}\frac{\mathrm{d}\theta}{\mathrm{d}T}=-D\cos\gamma \tag{2}$$

where m is the mass of the satellite, M the mass of the Earth, D the drag force, and γ the flight-path angle.

Now, D is given by

$$D=\tfrac{1}{2}\rho V^2SC_D \tag{3}$$

where ρ is the atmospheric density, S the cross-sectional area of the satellite, and C_D the constant drag coefficient. The magnitude of the velocity is

$$V=[(\mathrm{d}R/\mathrm{d}T)^2+R^2(\mathrm{d}\theta/\mathrm{d}T)^2]^{1/2} \tag{4}$$

and the flight-path angle γ is

$$\gamma=\tan^{-1}\frac{\mathrm{d}R/\mathrm{d}T}{R\mathrm{d}\theta/\mathrm{d}T} \tag{5}$$

The exponential density model can be written as

$$\rho=\rho_0e^{-(R-R_0)/H}\qquad \rho_0=\rho(R_0) \tag{6}$$

where R_0 is a reference radius and H the scale height of the atmosphere. Fitting Eq. (6) to any two values of ρ at two different altitudes fixes ρ_0 and H. In terms of the nondimensional parameters $r=R/R_0$, θ, and $t=T/(R_0^3/GM)^{1/2}$, the equations of motion become

$$\ddot{r}-r\dot{\theta}^2=-1/r^2-\epsilon e^{-(r-1)/H_S}\dot{r}(\dot{r}^2+r^2\dot{\theta}^2)^{1/2} \tag{7}$$

$$r\ddot{\theta}+2\dot{r}\dot{\theta}=-\epsilon e^{-(r-1)/H_S}r\dot{\theta}(\dot{r}^2+r^2\dot{\theta}^2)^{1/2} \tag{8}$$

where $H_S=H/R_0$, and $\epsilon=C_D\rho_0SR_0/2m$ and is small if ρ_0 is small. Transforming Eqs. (7) and (8) to $u(\theta)$ and $t(\theta)$ with $u=1/r$ then gives

$$u''+u-u^4t'^2=0 \tag{9}$$

$$(u^2t')'=\epsilon e^{-(1-u)/H_Su}t'(u'^2+u^2)^{1/2} \tag{10}$$

where u is a harmonic function of θ if $\epsilon=0^1$, and $(\ )'=\mathrm{d}/\mathrm{d}\theta$.

The initial conditions we adopt correspond to the satellite being at pericenter at $t=0$. Also, with no loss of generality, we may choose the argument of pericenter $\omega=0$. Hence, the initial $(\theta=0)$ conditions are

$$u(0)=1\qquad t(0)=0\qquad u'(0)=0\qquad t'(0)=\sigma<1 \tag{11}$$

where σ is the reciprocal angular velocity initially.

If $\epsilon=0$, the above initial conditions define a unique Keplerian ellipse with constant elements a, e, ω, and τ, where a, e, and τ are the semimajor axis, the eccentricity, and the time of passage through pericenter, respectively. With $\epsilon\neq0$ and to order unity, that is, considering the first term in a series expansion in powers of ϵ, the motion will still be in the form of a Keplerian orbit but with slowly varying elements. It is, therefore, convenient to express the initial conditions of Eq. (11) in terms of equivalent conditions on the initial values of a, e, ω, and τ.

As the satellite starts at the pericenter, $\tau(0)=0$ from Kepler's equation for the time history of the orbit. Moreover, we chose $\omega(0)=0$. Since the pericenter distance is $a(1-e)$, we have

$$a(0)[1-e(0)]=1 \tag{12}$$

Differentiating Kepler's equation for the time history with respect to θ and using the conventional definition for the eccentric anomaly gives

$$\sigma=1/\sqrt{1+e(0)} \tag{13}$$

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Index categories: Earth-Orbital Trajectories; Entry Vehicles and Landers.

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